On the Biological Foundation of Risk Preferences

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Ellsberg Paradox

Urn containing balls

- 33 red balls
- 66 white or black balls

Two choice problems:

C1. £1 if red   or   £1 if black

Expected Utility Model

- ω : number of black balls
- Decision Maker forms a belief over ω : G
- Takes expectation when evaluating a lottery

Evaluation of Lotteries

Recall C1. £1 if red   or £1 if black

If bet on red: probability of winning $33/99 = 1/3$.

If bet on black: probability of winning

$$\int_0^{66} \omega/99 dG(\omega) = \frac{\mathbb{E}_G[\omega]}{99}.$$

"Choose black if you believe that there are more black than red".

Paradox

C1. £1 if red or £1 if black

individuals choose red $\Rightarrow \mathbb{E}_G[\omega] < 33$

C2. £1 if red or £1 if white

individuals choose red $\Rightarrow \mathbb{E}_G[\omega] > 33$

Evolutionary Approach

- choice behavior is genetic
- individuals face choices over lotteries
- *premise:* only the fastest growing gene survives

Cohen-Robson Model

- state of the world: $\Omega \sim G$
- outcomes in offspring: X
- lottery: F_ω is a distribution over X.
- time is discrete
- lotteries are independent across time
- gene: a choice from lotteries

Main Result

The utility criterion is:

$$\int_{\Omega} \ln\left(\int x dF_{\omega}(x)\right) dG(\omega)$$

Growth Rate:

(continuously compounded growth rate)

y_t : population at time t

$$y_t \sim cg^t$$

$$\ln y_t \sim \ln c + t \ln g$$

$$\lim_{t \to \infty} \frac{\ln y_t}{t} = g$$

Proof

Let ω_t denote the realization of ω at t.

Let $\mu(\omega) = \int f(x)\, dF_\omega(x)$.

Let $y_0 = 1$.

Then:

$$y_T = y_0 \prod_{t=1}^{T} \frac{y_t}{y_{t-1}} = \prod_{t=1}^{T} \mu(\omega_t).$$

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So,

$$\lim_{T\to\infty} \frac{\log y_T}{T} = \lim_{T\to\infty} \frac{\sum \log \mu(\omega_t)}{T}.$$

By Birkhoff's Theorem:

$$\begin{aligned}
\lim_{T\to\infty} \frac{\sum \log \mu(\omega_t)}{T} &= \int \log \mu(\omega)\, dG(\omega) \\
&= \int \log \left[\int f(x)\, dF_\omega(x) \right] dG(\omega).
\end{aligned}$$

Observation:

There are discrete changes in the size of the population

Question:

What happens if the population evolves smoothly?

Continuous-time Model

Lottery: $\left(\Omega, G, \{F_e(\cdot|\omega)\}_{\omega\in\Omega}\right)$

Ω : set of states of the world

G : ergodic distribution on Ω

environmental shocks: $\{F_e(\cdot|\omega)\}_{\omega\in\Omega}$

- net birth-rate $\varepsilon_t \sim F_e(\varepsilon_t|\omega_t)$
- $\mathbb{E}(\varepsilon|\omega)$ is a bounded function of ω,
- $F_e(\varepsilon|\omega)$ is uniformly continuous in ω.
- $\varepsilon_t' s$ are independent across individuals conditional on ω_t.
- $r(\omega_t) \equiv \int \varepsilon dF_e(\varepsilon|\omega_t)$

Main Result

$$\lim_{t \to \infty} \left(\frac{\log y_t}{t} \right) = \int \int \varepsilon dF_e \left(\varepsilon | \omega \right) dG \left(\omega \right)$$

.

Proof

$$y_{t+\Delta} \approx y_t \left[1 + \Delta r\left(\omega_t\right)\right] \Rightarrow \dot{y}_t = y_t r\left(\omega_t\right)$$

solution: $y_t = \exp\left[\int_0^t r\left(\omega_s\right) ds\right]$

So:

$$\lim_{t \to \infty} \frac{\log y_t}{t} = \lim_{t \to \infty} \frac{\int_0^t r\left(\omega_s\right) ds}{t}$$

Birkhoff's Theorem:

$$\lim_{t \to \infty} \frac{\int_0^t r(\omega_s) ds}{t} = \lim_{t \to \infty} \frac{\int_0^t \int \varepsilon_s dF_e(\varepsilon|\omega_t) ds}{t} = \int \int \varepsilon dF_e\left(\varepsilon|\omega\right) dG\left(\omega\right)$$

Endogenous growth and utility representation

- $\mathcal{L}$: set of lotteries

- $(\{l_k\}_{k=1}^{n}, \{\mu_k\}_{k=1}^{n})$: *environment,* $l_k \subset \mathcal{L}$, $|l_k| < \infty$ and μ_k is the arrival rate of l_k

- c: gene, $\{l_1, .., l_n\} \rightarrow \mathcal{L}$, s.t. $c(l_k) \in l_k$

Theorem

For each $\mathcal{L}$ and for each $\left(\{l_k\}_{k=1}^n, \{\mu_k\}_{k=1}^n\right)$, the surviving gene, c^*, satisfies

$$c^*(l_k) \in \arg\max_{L \in l_k} U(L),$$

for all $k = 1, .., n$.

What is going on?

Take Robson's model and *shorten* the intervals:

- outcome of the lottery is H or L with prob. half
- reproduce H or L kids

Shortening the intervals to Δ:

- outcome of the lottery is H or L with prob. half
- reproduce H^{Δ} or L^{Δ} kids

If the risk is aggregate:

$$\log g(\Delta) = \frac{1}{2}\left(\log H^{\Delta} + \log L^{\Delta}\right)$$

and

$$\Delta \log g = \log g(\Delta)$$

So:

$$\log g = \frac{1}{2}(\log H + \log L).$$

If the risk is idiosyncratic:

$$g(\Delta) = \frac{H^{\Delta} + L^{\Delta}}{2}$$

and

$$\Delta \log g = \log g(\Delta),$$

so

$$\log g = \frac{\log\left(\frac{H^{\Delta}+L^{\Delta}}{2}\right)}{\Delta}.$$

Is it true that

$$\lim_{\Delta\to 0}\frac{\log\left(\frac{H^{\Delta}+L^{\Delta}}{2}\right)}{\Delta}=\frac{1}{2}(\log H+\log L)?$$

L'Hopital Rule:

$$\begin{aligned}\lim_{\Delta\to 0}\frac{\log\left(\frac{H^{\Delta}+L^{\Delta}}{2}\right)}{\Delta} &= \lim_{\Delta\to\infty}\frac{1}{\left(\frac{H^{\Delta}+L^{\Delta}}{2}\right)}\left(\frac{H^{\Delta}\log H+L^{\Delta}\log L}{2}\right)\\ &= \frac{1}{2}(\log H+\log L)\,.\end{aligned}$$

Conclusion

If choices affect growth smoothly then the optimal choice behavior has a *Neumenn-Morgenstern* representation.